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Uniform Rational Approximation on Subsets of $[0, \infty]$*

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1. INTRODUCTION

Let m and n be integers with $0 \leqslant m \leqslant n$, and let X be a closed subset of $[0, \infty)$ containing at least $m+n+2$ points. $\bar{X}$ will denote X if $m<n$ and X is bounded, and $\bar{X}$ will denote $X \cup \{\infty\}$ otherwise. Let $C_0(\bar{X}) = \{f \in C(\bar{X}): f(\infty)=0 \text{ if } \infty \in \bar{X}\}$ (thus if X is unbounded, then $\lim_{x \to \infty, x \in X} f(x)=0$. Also, for the case that $m=n$, the theory given here holds provided only that $\lim_{x \to \infty, x \in X} f(x)$ exists). For any $Y \subseteq [0, \infty]$ with $Y \cap [0, \infty)$ closed, define $R_n^m[Y] = \{R=P/Q\colon P(x) = p_0 + p_1 x + \cdots + p_m x^m \in \Pi_m,\ Q(x) = q_0 + q_1 x + \cdots + q_n x^n \in \Pi_n,\ Q>0$ on Y, $\max_{0 \leqslant j \leqslant n} |q_j| = 1$, P/Q is in lowest terms, and $\partial P \leqslant \partial Q$ if Y is unbounded$\}$. Here $\partial P =$ degree of P, Π_m is the set of all polynomials of degree $\leqslant m$ with real coefficients, and Y unbounded means either $\infty \in Y$ or Y is an unbounded subset of $[0, \infty)$. If $\infty \in Y$, we define $Q(\infty) = \lim_{x \to \infty} Q(x)$ and $R(\infty) = \lim_{x \to \infty} (P(x)/Q(x))$, and we observe that $R \in R_n^m[Y]$ implies that this last limit exists and is finite. Furthermore, the requirement that $Q>0$

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on Y will be taken to be satisfied by $Q(\infty)=\infty$ when Q is not a constant. We also define $\bar{R}_n^m[Y]=\{R\in R_n^m[Y]:\ q_n\geqslant 0\}$. Letting $\|h\|_Y=\sup_{y\in Y}|h(y)|$, we say $R^*\in R_n^m[Y]$ is a best approximation to $f\in C_0(Y)$ on Y from $R_n^m[Y]$ if $\|f-R^*\|_Y\leqslant\|f-R\|_Y$ for all $R\in R_n^m[Y]$ (and similarly for $\bar{R}_n^m[Y]$).

We observe that $\bar{R}_n^m[Y]=R_n^m[Y]$ if Y is unbounded. The reason for introducing $\bar{R}_n^m[Y]$ is that in the $m<n$ case, if a best approximation from $\bar{R}_n^m[\bar{X}]$ exists, then this approximation is also best on $X\cap[0,b]$ from $\bar{R}_n^m[X\cap[0,b]]$ for some real number b, so this approximation can be computed by working on a bounded set. This follows from the fact that the alternation characterization for a best approximation from $\bar{R}_n^m[Y]$ is the same for Y bounded as it is for Y unbounded. Neither of these facts is true if $\bar{R}_n^m$ is replaced by R_n^m (see [8] for a discussion in the special case of reciprocal polynomial approximation).

In the case $m=n$, if a best approximation on $\bar{X}=X\cup\{\infty\}$ from $\bar{R}_n^m[X\cup\{\infty\}]$ exists, then this approximation is also best on $(X\cap[0,b])\cup\{\infty\}$ from $\bar{R}_n^m[(X\cap[0,b])\cup\{\infty\}]$ for some real number b. This follows from the fact that the alternation characterization for a best approximation on $Y\cup\{\infty\}$ from $\bar{R}_n^m[Y\cup\{\infty\}]$ is the same for Y bounded as it is for Y unbounded. Neither of these facts is true if the point at ∞ is removed, since in the $m=n$ case (unlike the $m<n$ case), ∞ can be an essential extreme point; that is, an extreme point whose removal would change the approximation. In this case, we will show how a differential correction based algorithm can be used to directly compute approximations on $Z\cup\{\infty\}$ where Z is finite.

In Section 2 we give an (alternation) characterization theorem, a "zero in the convex hull" characterization, and a strong uniqueness theorem. In Section 3 we give a discretization theorem and examples.

We require some additional notation. Given $R^*=P^*/Q^*\in\bar{R}_n^m[\bar{X}]$, we define $d(R^*)=\min(m-\partial P^*,\ n-\partial Q^*)$ (we say R^* is nondegenerate if $d(R^*)=0$), $M(R^*)=\{x\in\bar{X}:\ |f(x)-R^*(x)|=\|f-R^*\|_{\bar{X}}\}$, and $\sigma(x)=\operatorname{sgn}(f(x)-R^*(x))$. We say $\{x_1,\ldots,x_N\}\subseteq M(R^*)$ with $x_1<x_2<\cdots<x_N$ is an alternating set of length N for $f-R^*$ if $f(x_{i+1})-R^*(x_{i+1})=-(f(x_i)-R^*(x_i))$ for $i=1,\ldots,N-1$. If N is minimal but sufficiently large to guarantee that R^* is a best approximation to f on $\bar{X}$ from $\bar{R}_n^m[\bar{X}]$ according to Theorem 2.2 then we call $\{x_1,\ldots,x_N\}$ an *alternant* for $f-R^*$. If $P\in\Pi_m$ and $\{P_k\}\subseteq\Pi_m$, $P_k\rightrightarrows P$ will mean that the coefficients of P_k converge to those of P (and similarly for $Q\in\Pi_n$ and $\{Q_k\}\subseteq\Pi_n$). Finally,

$$D(x)\equiv\begin{cases}(x+1)^n, & m=n\\ 1, & m<n.\end{cases}\tag{1.1}$$

Some of the results in this paper for the case where X is unbounded have been proved, in a somewhat different situation, in [1, 2].

2. Characterization and Uniqueness Results

We have, for approximating from $\bar{R}_n^m[\bar{X}]$,

THEOREM 2.1 (Kolmogoroff). *$R^* \in \bar{R}_n^m[\bar{X}]$ is a best approximation to $f \in C_0(\bar{X})$ iff*

$$\min_{x \in M(R^*)} (f(x) - R^*(x))(R(x) - R^*(x)) \leqslant 0, \qquad \forall R \in \bar{R}_n^m[\bar{X}].$$

THEOREM 2.2 (alternation and uniqueness). *Suppose $f \in C_0(\bar{X})$ and $R^* = P^*/Q^* \in \bar{R}_n^m[\bar{X}]$.*

(1) *If $m = n$, then R^* is a best approximation to f on $\bar{X}$ iff there exists an alternating set for $f - R^*$ in $\bar{X}$ of length $m + n + 2 - d(R^*)$;*

(2a) *if $m < n$ and $n - \partial Q^* \leqslant m - \partial P^*$, then R^* is a best approximation to f on $\bar{X}$ iff there exists an alternating set for $f - R^*$ in X of length $m + n + 2 - d(R^*)$;*

(2b) *if $m < n$ and $n - \partial Q^* > m - \partial P^*$, then R^* is a best approximation to f on $\bar{X}$ iff there exists an alternating set for $f - R^*$ in X of length $m + n + 1 - d(R^*)$, and the sign of $f - R^*$ at the largest point in this set equals the sign of the leading coefficient of P^*.*

Furthermore, in all cases best approximations are unique.

Remark. If in case (2b) the maximum length of any alternating set for $f - R^*$ is $m + n + 1 - d(R^*)$, then one can think of the restriction $q_n^* \geqslant 0$ as playing the role of another point in the alternant (as in [8]). If this restriction were removed, and X is bounded, then the approximation could be improved (X unbounded requires $q_n^* \geqslant 0$ since $Q^* > 0$ on X).

The proofs of Theorems 2.1 and 2.2 are omitted, since they involve only small modifications in the proofs of Theorems 1, 2 and 4 in [1].

We note that sometimes when no best approximation from $\bar{R}_n^m[\bar{X}]$ exists, a best approximation from $\tilde{\bar{R}}_n^m[\bar{X}]$ will exist, where $\tilde{\bar{R}}_n^m[\bar{X}]$ is $\bar{R}_n^m[\bar{X}]$ with the restriction removed that P/Q be in lowest terms. Specifically, the common factor in P and Q cannot be cancelled, otherwise the new denominator would be negative somewhre on $\bar{X}$. Algorithms such as those in [9] will occasionally produce such an approximation. A modified alternation theorem for approximation from $\tilde{\bar{R}}_n^m[\bar{X}]$ could be proved as in [5], but we do not pursue it in this paper. Note that if $X = [0, \infty)$ then a best approximation from $\bar{R}_n^m[\bar{X}]$ will always exist (see Theorem 3.1).

We observe that Theorem 2.2 holds regardless of whether X is bounded or unbounded. This is not true in the case $m < n$ if $\bar{R}_n^m[\bar{X}]$ is replaced by $R_n^m[\bar{X}]$, since if X is bounded, any best approximation from $R_n^m[\bar{X}]$ must possess an alternant of length $m+n+2-d(R^*)$ by the standard theory. This unification of the theory for X bounded and unbounded allows us to prove the following theorem.

THEOREM 2.3. *Let $f \in C_0(\bar{X})$ and*

(1) *Suppose $m=n$ and a best approximation R^* on $\bar{X}$ to f exists from $\bar{R}_n^m[\bar{X}]$. Then there is a real number b such that R^* is the best approximation on $(X \cap [0, b]) \cup \{\infty\}$ to f from $\bar{R}_n^m[(X \cap [0, b]) \cup \{\infty\}]$.*

(2) *Suppose $m<n$ and a best approximation R^* on $\bar{X}$ to f exists from $\bar{R}_n^m[\bar{X}]$. Then there is a real number b such that R^* is the best approximation on $X \cap [0, b]$ to f from $\bar{R}_n^m[X \cap [0, b]]$.*

Proof. We prove (2) for the case that R^* satisfies (2b) of Theorem 2.2. The other cases follow in a similar manner applying the alternation theory for best uniform nonconstrained rational approximations. Let $\{x_1, ..., x_{m+n+1-d(R^*)}\}$ be an alternant for $f-R^*$ in X, and let $b=x_{m+n+1-d(R^*)}$. Then $\{x_1, ..., x_{m+n+1-d(R^*)}\}$ is an alternant for $f-R^*$ in $X \cap [0, b] = (\overline{X \cap [0, b]})$ and $R^* \in \bar{R}_n^m[X \cap [0, b]]$, so R^* is best to f on $X \cap [0, b]$ by Theorem 2.2(2b). ∎

Although it is desirable to find a constructive way of choosing b (as in [8]), and such a method exists if $m=n$ and $X=[0, \infty)$, it could require the computation of as many as $4m+8$ rational approximations. Therefore, in most situations, one is better off just trying larger values for b until one is found which works. The fact that such a number b does exist shows that approximation on unbounded X can be done by approximating on a bounded subset (with the point at ∞ appended if $m=n$).

The reason for appending ∞ in the case $m=n$ is that Theorem 2.3 is false otherwise. To see this, construct an example (e.g., Example 2 in Section 3) where every alternating set of length $m+n+2-d(R^*)$ contains the point at ∞. Then R^* is not best on $X \cap [0, b]$ for any real b. For the $m<n$ case, ∞ cannot be an "essential" extreme point, since best approximations are characterized by a bounded alternant (e.g., Theorem 2.2).

The following two lemmas will be useful. We only sketch the proofs, since the arguments are similar to those in [2].

LEMMA 2.1. *Suppose X is a closed subset of $[0, \infty)$ containing at least $m+n+2$ points, and R^* is a best approximation to $f \in C_0(\bar{X}) \setminus \bar{R}_n^m[\bar{X}]$ from $\bar{R}_n^m[\bar{X}]$. Let $A=\{x_1, ..., x_N\} \subseteq \bar{X}$ be an alternant for $f-R^*$, and let $A_k = \{x_{1k}, ..., x_{Nk}\} \subseteq \bar{X}$ satisfy $x_{ik} \to x_i'$ for $i=1, ..., N$, where $x_1' < x_2' < \cdots < x_N'$ and $x_N' < \infty$ if $m<n$. Let $\{P_k\} \subseteq \Pi_m$, $\{Q_k\} \subseteq \Pi_n$, $\{\varepsilon_k\}$ satisfy*

$$P_k \rightrightarrows P \in \Pi_m, \qquad Q_k \rightrightarrows Q \in \Pi_n, \qquad \varepsilon_k \geqslant 0, \qquad \varepsilon_k \to 0,$$

where

$$P_k(x)=\sum_{j=0}^{m} p_{jk}x^j, \qquad P(x)=\sum_{j=0}^{m} p_jx^j,$$

$$Q_k(x)=\sum_{j=0}^{n} q_{jk}x^j, \qquad Q(x)=\sum_{j=0}^{n} q_jx^j.$$

Suppose that for all k, either

(i) $R_k = P_k/Q_k \in \bar{R}_n^m[A_k]$ *and* $\sigma(x_i)(R_k - R^*)(x_{ik}) \geqslant -\varepsilon_k$ *for* $i=1, ..., N$, *or*

(ii) $q_{nk} \geqslant 0$ *if* $N=m+m+1-d(R^*)$, *and* $\sigma(x_i)(P_k/D - R^*(Q_k/D))(x_{ik}) \geqslant -\varepsilon_k$ *for* $i=1, ..., N$.

Then $PQ^*-P^*Q\equiv 0$. *Furthermore, if* R^* *is nondegenerate,* $\max_{0\leqslant j\leqslant n}|q_{jk}|=1$, $\forall k$, *and* $\max_{0\leqslant i\leqslant N}Q_k(x_{ik})\geqslant 0$, $\forall k$, *then* $P=P^*$ *and* $Q=Q^*$, *so* $P_k \rightrightarrows P^*$ *and* $Q_k \rightrightarrows Q^*$.

Proof. We first observe that (i) implies (ii) (with a different $\{\varepsilon_k\}$) since if (i) holds, then for all sufficiently large k and for $i=1, ..., N$, we have

$$\begin{aligned}
&\sigma(x_i)\left(\frac{P_k}{D}-R^*\frac{Q_k}{D}\right)(x_{ik})\\
&=\sigma(x_i)\frac{Q_k}{D}(x_{ik})(R_k-R^*)(x_{ik})\\
&\geqslant -\frac{Q_k}{D}(x_{ik})\,\varepsilon_k\\
&\geqslant \begin{cases} -\dfrac{1}{D(x_i')-\frac{1}{2}}\left(\displaystyle\sum_{j=0}^{n}(x_i'+1)^j\right)\varepsilon_k \to 0 & \text{if} \quad x_i'<\infty\\ -(q_n+1)\varepsilon_k\to 0 & \text{if} \quad x_i'=\infty.\end{cases}
\end{aligned}$$

Thus we assume (ii) holds, and divide the proof into two parts.

Case 1. ($N=m+n+2-d(R^*)$). For $i=1, ..., N-1$ we have

$$\sigma(x_i)\left(\frac{P_kQ^*-P^*Q_k}{Q^*D}\right)(x_{ik})\geqslant -\varepsilon_k,$$

so $\sigma(x_i)(PQ^*-P^*Q)(x_i')\geqslant 0$.

If $x_N'<\infty$, then the last inequality holds for $i=N$ also. Thus, counting zeros implies that $PQ^*-P^*Q\equiv 0$. Suppose $x_N'=\infty$ (so $m=n$ by assumption) and $PQ^* - P^*Q \not\equiv 0$, then $\partial(PQ^* - P^*Q) = m + n - d(R^*)$ and this implies that

$$\left(\frac{P}{D} - R^* \frac{Q}{D}\right)(\infty) = \frac{PQ^* - P^*Q}{Q^*D}(\infty) \neq 0.$$

Thus, for some real $\tilde{x} > x'_{N-1}$, sufficiently large, we have

$$\sigma(x_N)\,\mathrm{sgn}(PQ^* - P^*Q)(\tilde{x})$$
$$= \sigma(x_N)\,\mathrm{sgn}\left(\frac{P}{D} - R^* \frac{Q}{D}\right)(\tilde{x}) = \sigma(x_N)\,\mathrm{sgn}\left(\frac{P}{D} - R^* \frac{Q}{D}\right)(\infty) > 0,$$

so again $PQ^* - P^*Q \equiv 0$, as desired.

The last sentence of the lemma now follows by standard arguments.

Case 2 ($N = m + n + 1 - d(R^*)$). As in Case 1, if $PQ^* - P^*Q \not\equiv 0$ then we must have $\partial(PQ^* - P^*Q) = m + n - d(R^*)$. Using Theorem 2.2, we have

$$\partial(PQ^*) \leqslant m + \partial Q^* < n + \partial P^* \leqslant m + n - d(R^*).$$

So again $\partial Q = n$, $\partial P^* = m - d(R^*)$ and hence $q_n > 0$. Thus for real $\tilde{x} > x'_N$ (sufficiently large) we have

$$\mathrm{sgn}(PQ^* - P^*Q)(\tilde{x}) = -\mathrm{sgn}(P^*Q)(\tilde{x})$$
$$= -\mathrm{sgn}(\text{leading coefficient of } P^*) = -\sigma(x_N),$$

so $-\sigma(x_N)\cdot(PQ^* - P^*Q)(\tilde{x}) > 0$, and the rest follows as in Case 1. ∎

LEMMA 2.2. *Suppose X is a closed subset of $[0, \infty)$, Y is a compact subset of X containing at least $m + n + 2$ points, $R^* \in \bar{R}_n^m[\bar{X}]$ is nondegenerate, and $\{P_k\} \subseteq \Pi_m$, $\{Q_k\} \subseteq \Pi_n$ satisfy $P_k \rightrightarrows P^*$ and $Q_k \rightrightarrows Q^*$. If $m < n$, suppose further that $\partial Q^* \geqslant n - 1$, $q_{nk} \geqslant 0$ for all k if $\partial Q^* = n - 1$, and either $\partial Q^* \geqslant m + 1$ or $q_{nk} = 0$ for all $k \geqslant$ some constant k_0. Then there exist constants Ω and $\in\, > 0$ such that for all k sufficiently large, $Q^* \geqslant \varepsilon$ and $Q_k \geqslant \varepsilon/2$ on $\bar{X}$, and $\|R_k - R^*\|_{\bar{X}} \leqslant \Omega\, \|R_k - R^*\|_Y$, where $R_k = P_k/Q_k$.*

Proof. If $m = n$, nondegeneracy implies $q_n^* > 0$. Assume X is unbounded; similar arguments work if X is bounded. Thus, regardless of whether $m = n$ or $m < n$, for all $k \geqslant$ some constant k_1 we will have either $q_{nk} \geqslant \frac{1}{2} q_n^* > 0$ (if $\partial Q^* = n$) or $q_{nk} \geqslant 0$, $q_n^* = 0$, $q_{n-1,k} \geqslant \frac{1}{2} q_{n-1}^* > 0$ (if $\partial Q^* = n - 1$). The lower bounds on Q^* and Q_k follow from this. If we let $(P_k Q^* - P^* Q_k)(x) = \sum_{l=0}^{m+n} a_{lk} x^l$ and consider the degrees of the numerator and denominator of $R_k - R^* = (P_k Q^* - P^* Q_k)/Q^* Q_k$, we also get $\|R_k - R^*\|_{\bar{X}} \leqslant$

$r_1 \max_{0 \leqslant l \leqslant m+n} |a_{lk}|$ for some constant r_1. Thus, if $Y \subseteq [0, L]$ for some $L > 0$, then for k sufficiently large we get (for some constant r_2), that

$$\|R_k - R^*\|_{\bar{X}} \leqslant r_1 r_2 \|P_k Q^* - P^* Q_k\|_Y = r_1 r_2 \|Q^* Q_k (R_k - R^*)\|_Y$$

$$\leqslant r_1 r_2 \cdot 2 \left(\sum_{j=0}^{n} L^j \right)^2 \|R_k - R^*\|_Y \equiv \Omega \|R_k - R^*\|_Y. \quad \blacksquare$$

One can prove the following "zero in the convex hull" characterization of best approximations in our setting. The proof, which uses Lemma 2.1 and arguments similar to those in [3], will be omitted.

THEOREM 2.4. *Given X a closed subset of $[0, \infty)$ with at least $m+n+2$ points, $f \in C_0[\bar{X}] \setminus \bar{R}_n^m[\bar{X}]$, and $R^* \in \bar{R}_n^m[\bar{X}]$, let $S_1 = \{[0, ..., 0, -1]\} \subseteq R^{m+n+2}$ if $m < n$ and $q_n^* = 0$, and $S_1 = \varnothing$ otherwise. Further let $M'(R^*) = M(R^*) \setminus [\bar{c}, \infty]$ (with $\bar{c} = \inf\{c: [c, \infty] \subseteq M(R^*)\}$) if $m < n$ and $\infty \in M(R^*)$, and $M'(R^*) = M(R^*)$ otherwise. Let $D(x)$ be defined by (1.1) and let*

$$S = \left\{ \sigma(x) \left[\frac{1}{D(x)}, \frac{x}{D(x)}, ..., \frac{x^m}{D(x)}, \frac{R^*(x)}{D(x)}, \frac{xR^*(x)}{D(x)}, ..., \frac{x^n R^*(x)}{D(x)} \right] : x \in M'(R^*) \right\} \cup S_1.$$

Then R^* is a best approximation to f from $\bar{R}_n^m[\bar{X}]$ on $\bar{X}$ iff $0 \in \mathscr{H}(S) \equiv$ the convex hull of S.

Next we prove a strong uniqueness theorem which we require later. The proof follows the line of argument used to obtain strong uniqueness results in [2, 3].

THEOREM 2.5. *Suppose X is a closed subset of $[0, \infty)$ containing at least $m+n+2$ points and $R^* \in \bar{R}_n^m[\bar{X}]$ is a best approximation to $f \in C_0(\bar{X})$. Suppose R^* is nondegenerate, and if $m < n$ also assume either X is bounded or $\partial Q^* \geqslant \max(n-1, m+1)$. Then there is a constant $\gamma > 0$ such that for all $R \in \bar{R}_n^m[\bar{X}]$,*

$$\|f - R\|_{\bar{X}} \geqslant \|f - R^*\|_{\bar{X}} + \gamma \|R - R^*\|_{\bar{X}}.$$

Proof. If $f \in \bar{R}_n^m[\bar{X}]$ the result follows immediately, so assume $f \notin \bar{R}_n^m[\bar{X}]$. Suppose (by way of contradiction) there exists $\{R_k\} \subseteq \bar{R}_n^m[\bar{X}]$ with $R_k \neq R^*$ for all k and

$$\gamma(R_k) \equiv \frac{\|f - R_k\|_X - \|f - R^*\|_X}{\|R_k - R^*\|_{\bar{X}}} \to 0.$$

Then $\|R_k\|_{\bar{X}}$ is bounded (otherwise $\gamma(R_k) \nrightarrow 0$), so using subsequences, if necessary, we can assume $P_k \rightrightarrows P \in \Pi_m$, $Q_k \rightrightarrows Q \in \Pi_n$. Let A be an alternant for $f - R^*$. For any $y \in A$, we have

$$\begin{aligned}\gamma(R_k)\,\|R_k - R^*\|_{\bar{X}} &= \|f - R_k\|_{\bar{X}} - \|f - R^*\|_{\bar{X}}\\ &\geqslant \sigma(y)(f - R_k)(y) - \sigma(y)(f - R^*)(y)\\ &= \sigma(y)(R^* - R_k)(y).\end{aligned}$$

By Lemma 2.1, $P \equiv P^*$ and $Q \equiv Q^*$, so $P_k \rightrightarrows P^*$ and $Q_k \rightrightarrows Q^*$. Now let $L > 0$ be such that $X \subseteq [0, L]$ if X is bounded, otherwise $X \cap [0, L]$ has at least $m+n+2$ points. In either case, define $Y = X \cap [0, L]$. Then by Lemma 2.2 there are constants $\varepsilon > 0$, k_0, and Ω such that for $k \geqslant k_0$ we have $Q^* \geqslant \varepsilon$ on $\bar{X}$, $Q_k \geqslant \varepsilon/2$ on $\bar{X}$, and $\|R_k - R^*\|_{\bar{X}} \leqslant \Omega\,\|R_k - R^*\|_Y$. Now let $(P_k Q^* - P^* Q_k)(x) = \sum_{l=0}^{m+n} a_{lk} x^l$, $\beta_k = \max_{0 \leqslant l \leqslant m+n} |a_{lk}|$, and $c = \inf_{k \geqslant k_0} \max_{y \in A} \sigma(y)((R^* - R_k)(y)/\beta_k)$. Then arguments similar to those in Lemma 2.1 can be used to show $c > 0$, by showing that assuming the contrary implies $\sum_{l=0}^{m+n}(a_{lk}/\beta_k)x^l$ converges to the zero polynomial. Now drawing subsequences if necessary, let y_0 be such that $\sigma(y_0)((R^* - R_k)(y_0)/\beta_k) \geqslant c$, for all $k \geqslant k_0$. For $k \geqslant k_0$ we have

$$\begin{aligned}\gamma(R_k)\,\|R_k - R^*\|_X &\geqslant \sigma(y_0)(R^* - R_k)(y_0) = \beta_k \sigma(y_0)\,\frac{(R^* - R_k)(y_0)}{\beta_k}\\ &\geqslant \beta_k c \geqslant \frac{\|P_k Q^* - P^* Q_k\|_Y}{\sum_{l=0}^{m+n} L^l}\cdot c\\ &= \frac{c}{\sum_{l=0}^{m+n} L^l}\,\|Q^* Q_k (R_k - R^*)\|_Y\\ &\geqslant \frac{c}{\sum_{l=0}^{m+n} L^l}\cdot \varepsilon \cdot \frac{\varepsilon}{2}\,\|R_k - R^*\|_Y\\ &\geqslant \frac{\varepsilon^2 c}{2\sum_{l=0}^{m+n} L^l}\cdot\frac{1}{\Omega}\,\|R_k - R^*\|_{\bar{X}},\end{aligned}$$

so $\gamma(R_k) \geqslant \varepsilon^2 c/(2\Omega \sum_{l=0}^{m+n} L^l)$, which violates $\gamma(R_k) \to 0$. ∎

3. Discretization Results, Computation and Examples

In actually computing approximations one normally works on a finite point set, so it is of some interest to know how such a computed approximation compares to the best approximation on $[0, \infty]$. The following discretization theorem sheds some light on this question.

THEOREM 3.1. *Suppose* $f \in C_0[0, \infty] \backslash \bar{R}_n^m[0, \infty]$.

(i) *A best approximation,* R_∞, *from* $\bar{R}_n^m[0, \infty]$ *on* $[0, \infty]$ *exists.*

(ii) *Suppose* R_∞ *is nondegenerate, and* b *is so large that* R_∞ *is also best on* $\overline{[0, b]}$. *Then a best approximation* R_Z *exists on* $\bar{Z}$ *from* $\bar{R}_n^m[\bar{Z}]$ *for all* $Z \subseteq [0, b]$ *with* $\|Z\| \equiv \sup_{x \in [0,b]} \inf_{y \in Z} |x-y|$ *sufficiently small, and* R_Z *converges uniformly to* R_∞ *on* $\overline{[0, b]}$ *as* $\|Z\| \to 0$. *Furthermore,* $\lim_{\|Z\| \to 0} \|f - R_Z\|_{\bar{Z}} = \|f - R_\infty\|_{\overline{[0,b]}}$.

(iii) *Under the hypothesis of* (ii), *suppose further that if* $m < n$, *then* $\partial Q_\infty \geqslant n-1$ *and either* $\partial Q_\infty \geqslant m+1$ *or* $f - R_\infty$ *has no alternant of length* $m+n+2$ *in* $[0, b]$. *Then* $R_Z \in \bar{R}_n^m[0, \infty]$ *for all* $\|Z\|$ *sufficiently small, and* R_Z *converges uniformly to* R_∞ *on* $[0, \infty]$ *as* $\|Z\| \to 0$. *Furthermore* $\lim_{\|Z\| \to 0} \|f - R_Z\|_Z = \|f - R_\infty\|_{[0,\infty]}$.

(iv) *Under the hypothesis of* (iii), *for* $\|Z\|$ *sufficiently small there is a constant* M_1 (*independent of* Z), *such that*

$$\|f - R_Z\|_{[0,\infty]} - \|f - R_\infty\|_{[0,\infty]} \leqslant M_1(\omega(\|Z\|) + \|Z\|),$$

where

$$\omega(\delta) \equiv \max\{|f(x) - f(y)| : x, y \in [0, \infty) \text{ and } |x-y| \leqslant \delta\}.$$

(v) *Under the hypothesis of* (iii), *assume also that* $0 \in Z$ *and* $b \in Z$, *and* f'' *is continuous on* $[0, b]$. *Then for* $\|Z\|$ *sufficiently small there is a constant* M_2 *such that*

$$\|f - R_Z\|_{[0,\infty]} - \|f - R_\infty\|_{[0,\infty]} \leqslant M_2 \|Z\|^2.$$

Proof. (i) This result (cited in [1]) comes from the work of Werner [10]. It can be proved using the standard existence proof for a bounded interval.

(ii) The third sentence of (ii) follows from the second; the second is proved by small modifications of the arguments in [4]. Lemma 2 of [4] is replaced by the following result, which follows from Lemma 2.1 of this paper by a contradiction argument. Let $\varepsilon > 0$ be given and $A = \{x_1, ..., x_N\} \subseteq \overline{[0, b]}$ be an alternant for $f - R_\infty$; then there exist $\delta > 0$ and a function $\eta(\varepsilon)$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that if $A' = \{x'_1, ..., x'_N\} \subseteq \overline{[0, b]}$ is fixed with $|x'_i - x_i| < \delta$ if $x_i < \infty$, and $x'_i = \infty$ if $x_i = \infty$ for $i = 1, ..., N$, and $R \in \bar{R}_n^m[A']$ satisfies $\sigma(x_i)(R - R_\infty)(x'_i) \geqslant -\varepsilon$ for $i = 1, ..., N$, then for $\varepsilon > 0$ sufficiently small we have $R \in \bar{R}_n^m\overline{[0, b]}$ and $\|R - R_\infty\|_{\overline{[0,b]}} \leqslant \eta(\varepsilon)$.

(iii) We first observe that if $\partial Q_\infty = n-1 = m$ and $f - R_\infty$ has no alternant of length $m+n+2$ in $[0, b]$ (note that $m < n$ so $\overline{[0, b]} = [0, b]$), then $\partial Q_Z = n-1$ for all Z with $\|Z\|$ sufficiently small. If this were not true, then considering a sequence $\{Z_k\}$ with $Z_k \subseteq [0, b]$, $\|Z_k\| \to 0$, R_k best on Z_k, and $\partial Q_k = n$ for all k, and (as in [2]) considering an accumulation point of alternants for $f - R_k$ on $[0, b]$, one can show that this accumulation point forms an alternant of length $m+n+2$ for $f - R_\infty$ in $[0, b]$, contrary to our assumption. Now it follows from Lemma 2.2 that there is a constant Ω such that for $\|Z\|$ sufficiently small, $R_Z \in \bar{R}_n^m[0, \infty]$ and $\|R_Z - R_\infty\|_{[0,\infty]} \leqslant \Omega \|R_Z - R_\infty\|_{[0,b]}$, so the uniform convergence on $[0, \infty]$ follows from (ii).

(iv) Using Lemma 2.2 and Theorem 2.5, there are constants Ω and $\gamma > 0$ such that for $\|Z\|$ sufficiently small we have

$$\|f - R_Z\|_{[0,\infty]} - \|f - R_\infty\|_{[0,\infty]} \leqslant \|R_Z - R_\infty\|_{[0,\infty]} \leqslant \Omega \|R_Z - R_\infty\|_{[0,b]} \leqslant \frac{\Omega}{\gamma} [\|f - R_Z\|_{[0,b]} - \|f - R_\infty\|_{[0,b]}],$$

so it suffices to show that

$$\|f - R_Z\|_{[0,b]} - \|f - R_\infty\|_{[0,b]} \leqslant \omega(\|Z\|) + M_3 \|Z\|$$

for some constant M_3 independent of Z. For $\|Z\|$ small, suppose $x \in [0, b]$ satisfies $|f(x) - R_Z(x)| = \|f - R_Z\|_{[0,b]}$, and then choose $y \in Z$ such that $|x - y| \leqslant \|Z\|$. Since $Q_\infty \geqslant \varepsilon$ on $[0, b]$ for some $\varepsilon > 0$, we must have $Q_Z \geqslant \varepsilon/2$ on $[0, b]$ for all $\|Z\|$ sufficiently small. Using this and the fact that the coefficients of P_Z and Q_Z are bounded, we have

$$\begin{aligned}
\|f - R_Z\|_{[0,b]} &= |f(x) - R_Z(x)| \\
&\leqslant |f(x) - f(y)| + |f(y) - R_Z(y)| \\
&\quad + \frac{|P_Z(y)\, Q_Z(x) - P_Z(x)\, Q_Z(y)|}{Q_Z(y)\, Q_Z(x)} \\
&\leqslant \omega(\|Z\|) + \|f - R_Z\|_Z \\
&\quad + \frac{4}{\varepsilon^2} |P_Z(y)\, Q_Z(x) - P_Z(y)\, Q_Z(y) \\
&\quad + P_Z(y)\, Q_Z(y) - P_Z(x)\, Q_Z(y)| \\
&\leqslant \omega(\|Z\|) + \|f - R_\infty\|_Z \\
&\quad + \frac{4}{\varepsilon^2}\left[|P_Z(y)| \left| \sum_{j=1}^{n} q_{jZ}(x^j - y^j) \right| + |Q_Z(y)| \right. \\
&\quad \left. \cdot \left| \sum_{i=1}^{m} p_{iZ}(y^i - x^i) \right| \right]
\end{aligned}$$

$$\leqslant \omega(\|Z\|) + \|f - R_\infty\|_{[0,b]}$$
$$+\frac{4\,|x-y|}{\varepsilon^2}\Bigg[|P_Z(y)| \sum_{j=1}^{n} |q_{jZ}(x^{j-1}+x^{j-2}y+\cdots+y^{j-1})|$$
$$+|Q_Z(y)| \sum_{i=1}^{m} |p_{iZ}(y^{i-1}+xy^{i-2}+\cdots+x^{i-1})|\Bigg]$$
$$\leqslant \omega(\|Z\|) + \|f - R_\infty\|_{[0,b]}$$
$$+\frac{4\,\|Z\|}{\varepsilon^2}\Bigg[\left(\sum_{i=0}^{m} |p_{iZ}|\,b^i\right)\left(\sum_{j=1}^{n} |q_{jZ}|\,jb^{j-1}\right)$$
$$+\left(\sum_{j=0}^{n} |q_{jZ}|\,b^j\right)\left(\sum_{i=1}^{m} |p_{iZ}|\,ib^{i-1}\right)\Bigg]$$
$$\leqslant \omega(\|Z\|) + \|f - R_\infty\|_{[0,b]} + M_3\,\|Z\|$$

for some constant M_3 independent of Z, and the resut follows.

(v) Arguing as in (iv), it suffices to show that

$$\|f - R_Z\|_{[0,b]} - \|f - R_\infty\|_{[0,b]} \leqslant M_4\,\|Z\|^2$$

for some constant M_4 independent of Z, with $\|Z\|$ sufficiently small. But this was shown in [6] using the results of Ellacott and Williams [7]. ∎

A natural question to ask at this point is: If b was chosen sufficiently large, does $\|Z\|$ sufficiently small guarantee that R_Z is best on $Z \cup [b, \infty]$? Under the assumptions of Theorem 3.1, part (iii), the answer is yes if $\infty \notin M(R_\infty)$, since then we can choose b so large that for all $x \geqslant b$, $|f(x) - R_\infty(x)| \leqslant \|f - R_\infty\|_{[0,\infty]} - \varepsilon_1$ for some $\varepsilon_1 > 0$, and use the fact that R_Z converges uniformly to R_∞ on $[0, \infty]$. The following example shows, however, that if $\infty \in M(R_\infty)$ it is possible that for any real $b > 0$ there exists $Z_b \subseteq [0, b]$ with $\|Z_b\|$ arbitrarily small and R_{Z_b} is not best on $Z_b \cup [b, \infty]$.

EXAMPLE 1. Let $f \in C_0[0, \infty]$ have values $-1/2$, $5/3$, $-1/6$, $21/11$, $-1/18$, $53/27$ and 0 at 0, 1, 2, 3, 4, 5 and 6, respectively. Define f to be linear between these points and define $f(x) = 0$ for $x \geqslant 6$. Then $R_\infty \in \bar{R}_3^2[0, \infty]$ defined by $R_\infty(x) = (1 + x^2)/(2 + x^2)$ is a best rational approximation to f on $[0, \infty]$ from $\bar{R}_3^2[0, \infty]$, with error norm 1 and alternant $\{0, 1, 2, 3, 4, 5\}$. Choose any b with $b > 5$; then R_∞ is best on $[0, b]$. For any positive integer k, define $R_k \in \bar{R}_3^2[0, \infty]$ by

$$R_k(x) = \frac{1 + (1/k)x + (1 - 1/k)x^2}{2 + x^2}.$$

Using elementary calculus, R_k has a unique maximum on $[0, \infty]$ at $\alpha_k = k - 2 + \sqrt{(k-2)^2 + 2}$, with $\beta_k = R_k(\alpha_k) = 1 - 1/k + O(1/k^2)$. Let k be so large that $\alpha_k > b$, and $|(f - R_k)(6)| < \beta_k - 1/k$. Now using the facts that, for large k, $|(f - R_k)(i)| > \beta_k - 1/k$ for $i = 0, ..., 5$ and $|f'(x)| > |R_k'(x)| + 19/12$ for $x \in (i, i+1)$, $i = 0, ..., 5$, we can construct

$$Z_k = [\delta_{0k}, 1 - \bar{\delta}_{1k}] \cup [1 + \delta_{1k}, 2 - \bar{\delta}_{2k}] \cup \cdots \cup [5 + \delta_{5k}, b]$$

with $\delta_{0k} \to 0^+, ..., \delta_{5k} \to 0^+$, $\bar{\delta}_{1k} \to 0^+, ..., \bar{\delta}_{5k} \to 0^+$ (so $\|Z_k\| \to 0$), R_k is best on Z_k with error norm $\beta_k - 1/k$ and alternant $\{\delta_{0k}, 1 + \delta_{1k}, ..., 5 + \delta_{5k}\}$, but R_k is not best on $Z_k \cup [b, \infty]$ since $\|f - R_k\|_{Z_k \cup [b, \infty]} = \beta_k$.

For numerical computation we use a combined First Remes-differential correction program [9], which computes approximations of the form

$$\frac{P(x)}{Q(x)} = \frac{p_0\phi_0(x) + \cdots + p_m\phi_m(x)}{q_0\psi_0(x) + \cdots + q_n\psi_n(x)}$$

on a finite set, with $|q_j| \leqslant 1$ for $j = 0, ..., n$ and $Q > 0$ on the set. Minor changes were made in two subroutines to force $0 \leqslant q_n \leqslant 1$ instead of $-1 \leqslant q_n \leqslant 1$. If $m < n$, we take $\phi_i(x) = x^i$ for $i = 0, ..., m$ and $\psi_j(x) = x^j$ for $j = 0, ..., n$. If $m = n$ we wish no compute an approximation on $Z \cup \{\infty\}$, where Z is a finite subset of $[0, \infty)$. In this case, we define

$$\phi_i(x) = \begin{cases} x^i, & x \in Z \\ 0, & x = \infty, i < m; \\ 1, & x = \infty, i = m \end{cases} \qquad \psi_j(x) = \begin{cases} x^j, & x \in Z \\ 0, & x = \infty, j < n \\ 1, & x = \infty, j = n \end{cases}$$

and thus $(P/Q)(\infty) = p_m/q_n$. If $d(R) > 0$, so $q_n = 0$, the program can still find an approximation of the form $\alpha(x)\, P(x)/(\alpha(x)\, Q(x))$, where $\alpha \in \Pi_{d(R)}$ is positive on $Z \cup \{\infty\}$, so the coefficient of x^n in the denominator will be positive.

EXAMPLE 2. Let $Z = \{0, 0.1, 0.2, ..., 20\}$. We approximated f on $Z \cup \{\infty\}$ from $\bar{R}_1^1[Z \cup \{\infty\}]$, where f takes the values -1, $-5/2$ and 0 at 0, 2 and 5, respectively, f is linear between these points, and $f(x) = 0$ for $x \geqslant 5$. To allow use of the program described above without further modification, we let 20.1 play the role of ∞. The computed approximation on $Z \cup \{\infty\}$ was

$$R(x) = \frac{-2 + 0.1x}{1 + 0.1x}$$

with error norm 1, achieved at $0^+, 2\ , 5^+$ and ∞ (where the sign indicates the sign of $f - R$). This approximation is best on $[0, \infty]$.

For comparison, we also computed the best approximation on $\{0, 0.1, 0.2, ..., 100\}$ (∞ not included); the result was $(-1.99385+0.11494x)/(1+0.08559x)$ with error norm 0.99385, achieved at 0^+, 2^-, 5^+ and 100^-. This approximation (unlike the previous one) is not best on $\{0, 0.1, 0.2, ..., 100\} \cup \{\infty\}$ as the error at ∞ is -1.34293.

Further details of proofs in this paper can be obtained from the authors.

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